Week 3 - Monday

COMP 4500

Last time

- What did we talk about last time?
- Survey of running times
- Worked exercises

Questions?

Logical warmup

- Three men in a cafe order food totaling $15
- They each contribute $5 to the bill
- The waiter takes the money to the manager who recognizes the three as friends and asks the waiter to return $5 to the men
- However, the waiter is dishonest
- Instead of splitting the $5 evenly (and who would get the last penny, anyway?), he gives each man $1 and pockets the remaining $2
- Now, each diner effectively paid $4
- Thus, the amount paid is $12
- Add the $2 in the waiter's pocket and the total comes to $14
- Where has the other $1 gone from the original $15?

Math practice

- Algebra practice:
  - https://www.khanacademy.org/math/algebra
- Algebra 2 practice:
  - https://www.khanacademy.org/math/algebra2
- Specifically, logarithms:
  - https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:logs
- Summation notation (stop before Riemann sums):
  - https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-3/v/sigma-notation-sum

Mathematical Induction

Induction

- General **induction** is moving from a specific set of facts to a general conclusion
- Example:
  - There are no tigers here.
  - I have a rock in my pocket.
  - Conclusion: My rock keeps tigers away.
- Induction can lead you to invalid conclusions
- In our previous proofs, we have used **deduction**, which reasons from general truths to a specific conclusion

Mathematical induction

- Mathematical induction is special
- First, we need a property ***P***(***n***) that's defined for integers ***n***
- Then, we need to know that it's true for some specific ***P***(***a***)
- Then, we try to show that for all integers $k \geq a$, if ***P***(***k***) is true, it must be the case that ***P***(***k***+1) is true
- If we do that, ***P***(***n***) is true for all integers $n \geq a$
- Why?

Proof by mathematical induction

- To prove a statement of the following form:
  - $\forall \boldsymbol{n} \in \mathbf{Z}$, where $\boldsymbol{n} \geq \boldsymbol{a}$, property $\boldsymbol{P}(\boldsymbol{n})$ is true
- Use the following steps:
  1. Basis Step: Show that the property is true for $\boldsymbol{P}(\boldsymbol{a})$
  2. Induction Step:
     - Suppose that the property is true for some $\boldsymbol{n} = \boldsymbol{k}$, where $\boldsymbol{k} \in \mathbf{Z}$, $\boldsymbol{k} \geq \boldsymbol{a}$
     - Now, show that, with that assumption, the property is also true for $\boldsymbol{k} + 1$

Example

- Prove that, for all integers $\boldsymbol{n} \geq 1$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- **Hint:** Use induction

Example

- Prove that, for all integers $n \geq 0$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

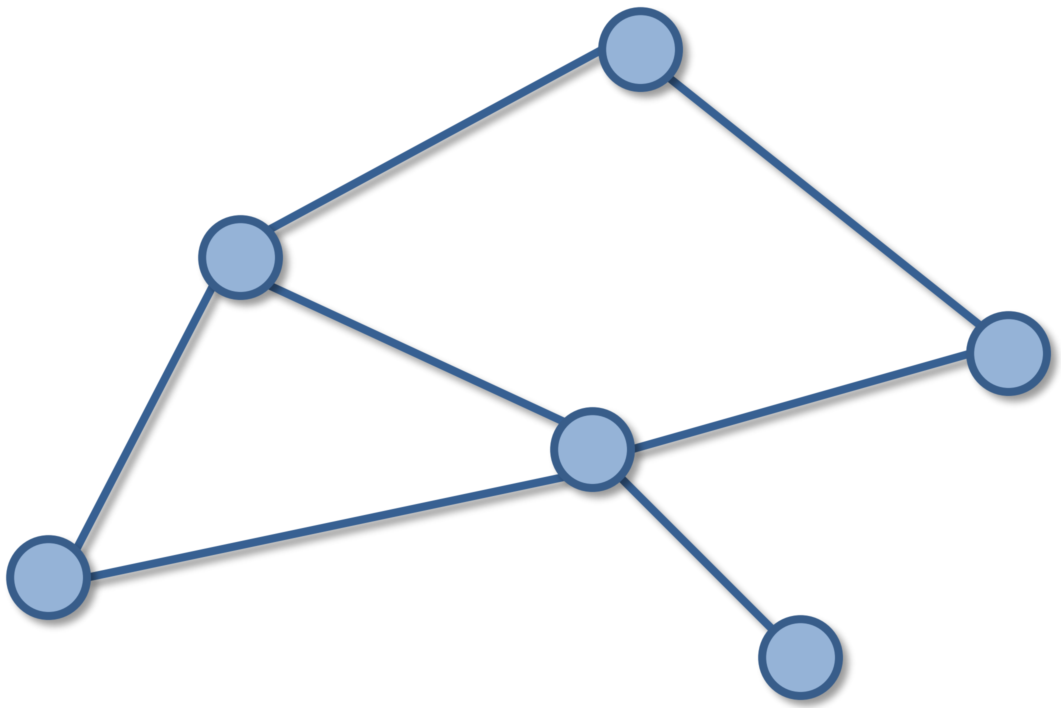
- **Hint:** Use induction

Three-Sentence Summary of Graph Definitions

Graph Definitions

What is a graph?

- Vertices (Nodes)
- Edges



Directed and undirected

- A regular (undirected) graph is one where edges represent a symmetrical relationship
  - For example, being siblings
- A **directed** graph is one where edges represent an asymmetrical relationship
  - For example, being taller than someone
- Directed graphs are usually drawn with arrows to show the direction of the relationship

Graph terminology

- A **graph** ***G*** is made up of two finite sets
  - **Vertices**: ***V***
  - **Edges**: ***E***
- Each edge is connected to either one or two vertices called its **endpoints**
- An edge with a single endpoint is called a **loop**
- Two edges with the same sets of endpoints are called **parallel**
- Edges are said to **connect** their endpoints
- Two vertices that share an edge are said to be **adjacent**
- The number of edges connected to a vertex is the **degree** of that vertex
- A graph with no edges is called **empty**

Graph representation

- We can represent graphs in many ways
- One is simply by listing all the vertices, all the edges, and all the vertices connected by each edge
- Let $\boldsymbol{V} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$
- Let $\boldsymbol{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7\}$ where each edge is specified in the table to the right
- Draw this graph
- Even for undirected graphs, the vertices making up an edge are often listed as ordered pairs: $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ instead of $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$

Edge	Vertices
$\boldsymbol{e}_1$	$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$
$\boldsymbol{e}_2$	$\{\boldsymbol{v}_1, \boldsymbol{v}_3\}$
$\boldsymbol{e}_3$	$\{\boldsymbol{v}_1, \boldsymbol{v}_3\}$
$\boldsymbol{e}_4$	$\{\boldsymbol{v}_2, \boldsymbol{v}_3\}$
$\boldsymbol{e}_5$	$\{\boldsymbol{v}_5, \boldsymbol{v}_6\}$
$\boldsymbol{e}_6$	$\{\boldsymbol{v}_5\}$
$\boldsymbol{e}_7$	$\{\boldsymbol{v}_6\}$

Drawing graphs

- Graphs can (generally) be drawn in many different ways
- We can label graphs to show that they are the same
- These two graphs are **isomorphic**, meaning that they can be labeled to be exactly the same

Graph Applications

Examples of graphs

- Transportation networks
- Communication networks
- Information networks
- Social networks
- Dependency networks

Transportation networks

- Vertices could be airports, cities, intersections
- Edges could be flights, highways, roads
- Edges are often undirected, but directed edges can represent one-way streets or similar
- Graphs of air travel tends to show a few hubs with many edges
- Street intersections or train stations tend to have few edges

Communication networks

- Vertices could be single computers or even all the computers served by a single ISP
- Edges are network connections between vertices
- In wireless networks, edges might exist between any nodes that are close enough
- Note that wireless networks can have directed edges, since some nodes might have a stronger transmitter

Information networks

- Vertices can be sources of information, such as books, papers, or websites
- Edges are references between those sources
- The World Wide Web is a directed graph where hyperlinks form the edges between pages
  - A page with many links pointing at it is popular, an important attribute that Google uses when ranking search results

Social networks

- Vertices can be people or perhaps businesses
- Edges can represent friendship, enmity, romantic relationships, financial relationships
- Friendship is often (but not necessarily) an undirected edge, but directed edges are common in such networks
- Networks can also show bipartite graphs between people and organizations

Dependency networks

- Vertices could be tasks or courses
- Edges could be dependencies between tasks or prerequisites for courses
  - Such edges are almost always directed
- A food web (who eats whom) is another possible application
  - Directed, but not necessarily acyclic!
  - You can eat bears, but they can eat you

Upcoming

Next time…

- Graph connectivity and traversal
- Implementing graph traversal with queues and stacks
- Testing bipartiteness

Reminders

- Read sections 3.2 and 3.3